1. Find the solution $y(x)$ of the two-point boundary value problem (15%)

$$y'' - 25y = 20xe^{-5x},\ y(0) = 0,\ y'(1) = 1$$

2. For a system described by the following equations (20%)

$$\frac{dx}{dt} + x + y = e^{-3t},\ x(0) = 1$$

$$\frac{dy}{dt} + 4x + y = 0,\ y(0) = 0$$

(a) Find the Laplace transform $X(s)$ of $x(t)$. Be sure to eliminate $Y(s)$ from the expression.

(b) Determine the final value of $x(t)$ as $t \to \infty$.

(c) Find the solution $y(t)$.

3. A farmer fills his silo with chopped corn. The entire corn plant (leaves, stem, and ear) is cut up into small pieces and blow into the top of the cylindrical silo at a rate $W_0$. The process is sketched in Figure 1. The diameter of the silo is $D$ and its height is $H$. The density of the chopped corn in the silo varies with the depth of the bed. The density $\rho$ at a point that has $z$ feet of material above it is

$$\rho(z) = \rho_0 + \beta z$$

where $\rho_0$ and $\beta$ are constants. (15%)

(a) Write the equations that describe the system and show how the height of the bed $h(t)$ varies as a function of time.

(b) What is the total weight of corn fodder that can be stored in the silo?

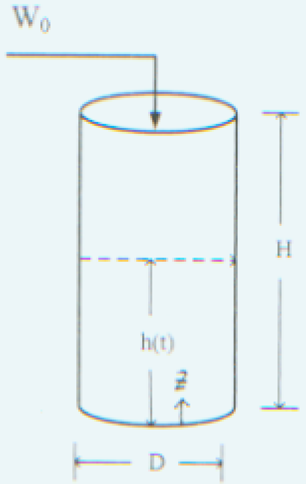


Figure 1. The chopped corn is filled into the silo.

4. Show that (i) div ($\mathbf{u} \times \mathbf{v}$) = $\mathbf{v}\cdot$curl $\mathbf{u}$ - $\mathbf{u}\cdot$curl $\mathbf{v}$ (5%)

(ii) div (curl $\mathbf{v}$) = 0 (5%)

5. Write down the form and application of the divergence theorem. (10%)

6. Circular membranes occur in drums, microphones, telephones, and so on. Its vibrations are governed by the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

where $u$ is deflection of the membrane, $c$ is a constant, $r$ is the distance to the center, and $\Theta$ is the counterclockwise angle from the $x$-axis. The membrane is fixed along its boundary $r$ = R. The initial deflection and velocity of the membrane are $f(r)$ and $g(r)$, respectively. Solve this partial differential equation. (30%)