[1]. (25 points) Γ(n)為 Gamma 函數，其形式定義如下：

$$\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}dx$$

①請證明下列循環公式

$$\Gamma(n+1)=n\Gamma(n) \quad , \quad n\in R \quad n>0$$

$$\Gamma(k+1)=k! \quad , \quad k\in N$$

②利用上述公式求

$$\int_0^\infty x^3e^{-x}dx=?$$

[2]. (25 points) 請解下列方程式

① $y''+2xy'=x^3 \qquad y(0)=0 \ , \ y'(0)=1$

② $x\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial t}=xt$ ，且 $\begin{cases} u(x,0)=0, & x\geq 0 \\ u(0,t)=0, & t\geq 0 \end{cases}$

[3]. A system can be described by the following two equations:

$$\frac{dx}{dt}=2x-3y, \quad x(0)=8$$

$$\frac{dy}{dt}=-2x+y, \quad y(0)=3$$

① (5 points) Find the Laplace transform $X(s)$ of $x(t)$. Be sure to eliminate $Y(s)$ from this expression.

② (10 points) Find the solution $y(t)$.

[4]. ① (5 points) Write down the Cauchy-Riemann equation for an analytic function $f(Z) = u(x, y) + i \cdot v(x, y)$, where $Z = x + i \cdot y$.

② (10 points) If $x^3 + 3x^2y - 3xy^2 - y^3$ is the real part of $f(Z)$. Determine the imaginary part of $f(Z)$.

[5]. Set $\mathbf{A} = \begin{bmatrix} -15 & -14 & -40 \\ 6 & 7 & 14 \\ 5 & 4 & 14 \end{bmatrix}$, and $f(x) = x^6 - 6x^5 + 12x^4 - 12x^3 + 12x^2 - 8x + 3$.

① (10 points) Find the eigenvalues of $\mathbf{A}$.

② (10 points) Find $f(\mathbf{A})$ by the Sylvester theorem.