

- [1]. (10 points) Find the particular solution of the first-order ordinary differential equation by the method of separation of variables.

$$xy' = y + 4x^5 \cos^2(y/x), \quad y(2) = 0.$$

- [2]. (10 points) Find a general solution of the second-order Euler-Cauchy equation.

$$4x^2 y'' + 4xy' - y = 0.$$

- [3]. (10 points) Find a general solution of the fourth-order ordinary differential equation with constant coefficients.

$$16y^{(4)} - 8y'' + y = 0.$$

- [4]. (10 points) Find a general solution of the second-order ordinary differential equation in terms of Bessel functions by using the following substitutions: $y = x^{-3}u$ and $2x = z$.

$$xy'' + 7y' + 4xy = 0.$$

- [5]. (10 points) Find the particular solution of the second-order differential equation containing the Dirac's delta function.

$$\begin{aligned} y'' + 2y' - 3y &= 100\delta(t-2) + 100\delta(t-3), \\ y(0) &= 1, \quad y'(0) = 0. \end{aligned}$$

- [6] (10 points) (a) Prove Cramer's Rule

If $\det A$ is not zero, then $A\mathbf{x} = \mathbf{b}$ has the unique solution

$$x_1 = \frac{\det B_1}{\det A}, x_2 = \frac{\det B_2}{\det A}, \dots, x_n = \frac{\det B_n}{\det A},$$

where matrix B_j is obtained by replacing the j th column of A by the vector \mathbf{b} . A is an $n \times n$ matrix.

- (10 points) (b) Use Cramer's Rule to solve

$$-4x_1 + x_3 = 0$$

$$-3x_1 + x_2 = 0$$

$$x_1 + x_2 + x_3 = 1$$

- [7] (10 points) Suppose that
- $\mathbf{F}(x, y) = f(x, y)\mathbf{i} + g(x, y)\mathbf{j}$
- is a continuous vector

function. Then $\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint_D \left[\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right] dA$, where C is a simple closed positively

oriented piecewise-smooth curve in the plane given by the position vector, \mathbf{R} .

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{R}$ if $\mathbf{F}(x, y) = [2y - \sin(x^2)]\mathbf{i} + [\exp^y - x]\mathbf{j}$ and C is the circle with

the center at $(0, 0)$ and a radius of 1.

- [8] (20 points) Determine the temperature distribution
- $u(x, t)$
- in a thin homogeneous bar of length
- L
- , given the initial temperature distribution throughout the bar at time zero (
- $f(x, t = 0) = 1$
-), if the ends are maintained at zero temperature for all time. The boundary value problem is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (0 < x < L, t > 0)$$

$$u(0, t) = u(L, t) = 0 \quad (t > 0)$$

$$u(x, 0) = f(x, t = 0) = 1$$

[Hint: Use separation of variables method to seek the solution. Fourier series of f on $[-L, L]$ is $a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$. The Fourier coefficients are

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx, \text{ and } b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx]$$