[1]. (10 points) Find the particular solution of the first-order ordinary differential equation by the method of separation of variables.

$$xy' = y + 4x^5 \cos^2(y/x), \quad y(2) = 0.$$

[2]. (10 points) Find a general solution of the second-order Euler-Cauchy equation.

$$4x^2 y'' + 4xy' - y = 0.$$

[3]. (10 points) Find a general solution of the fourth-order ordinary differential equation with constant coefficients.

$$16y^{iv} - 8y'' + y = 0.$$

[4]. (10 points) Find a general solution of the second-order ordinary differential equation in terms of Bessel functions by using the following substitutions: $y = x^{-3}u$ and $2x = z$.

$$xy'' + 7y' + 4xy = 0.$$

[5]. (10 points) Find the particular solution of the second-order differential equation containing the Dirac's delta function.

$$y'' + 2y' - 3y = 100\delta(t-2) + 100\delta(t-3),$$
$$y(0) = 1, \quad y'(0) = 0.$$

[6] **(10 points)** (a) Prove Cramer's Rule

If $\det A$ is not zero, then $A\mathbf{x} = \mathbf{b}$ has the unique solution

$x_1 = \dfrac{\det B_1}{\det A}, x_2 = \dfrac{\det B_2}{\det A}, \cdots x_n = \dfrac{\det B_n}{\det A}$, where matrix B_j is obtained by replacing the jth column of A by the vector b. A is an $n \times n$ matrix.

(10 points) (b) Use Cramer's Rule to solve

$$-4x_1 + x_3 = 0$$
$$-3x_1 + x_2 = 0$$
$$x_1 + x_2 + x_3 = 1$$

[7] **(10 points)** Suppose that $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$ is a continuous vector function. Then $\oint_C \mathbf{F} \cdot d\mathbf{R} = \iint \left[\dfrac{\partial g}{\partial x} - \dfrac{\partial f}{\partial y}\right] dA$, where C is a simple closed positively oriented piecewise-smooth curve in the plane given by the position vector, **R**.

Evaluate $\oint_C \mathbf{F} \cdot d\mathbf{R}$ if $\mathbf{F}(x,y) = \left[2y - \sin(x^2)\right]\mathbf{i} + \left[\exp^y - x\right]\mathbf{j}$ and C is the circle with the center at (0,0) and a radius of 1.

[8] **(20 points)** Determine the temperature distribution $u(x,t)$ in a thin homogeneous bar of length L, given the initial temperature distribution throughout the bar at time zero ($f(x,t=0)=1$), if the ends are maintained at zero temperature for all time. The boundary value problem is

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \quad (0<x<L,\ t>0)$$
$$u(0,t) = u(L,t) = 0 \quad (t>0)$$
$$u(x,0) = f(x,t=0) = 1$$

[Hint: Use separation of variables method to seek the solution. Fourier series of f on [-L, L] is $a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\dfrac{n\pi x}{L}\right) + b_n \sin\left(\dfrac{n\pi x}{L}\right)\right]$. The Fourier coefficients are

$a_0 = \dfrac{1}{2L}\int_{-L}^{L} f(x)dx, a_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\cos\left(\dfrac{n\pi x}{L}\right)dx$, and $b_n = \dfrac{1}{L}\int_{-L}^{L} f(x)\sin\left(\dfrac{n\pi x}{L}\right)dx$]