

1. Reduce the **Bernoulli equations** $y' + p(x)y = g(x)y^a$ (a any real number) to linear form. (10 points)
2. If y_1 is a solution of the differential equation $y'' + p(x)y' + q(x)y = 0$, find the second linearly independent solution. (10 points)
3. Sturm and Liouville have developed a theory of equations $[r(x)y']' + [q(x) + \lambda p(x)]y = 0$, $p(x) > 0$ that leads to practically useful series developments in terms of particular solutions on a given interval $a \leq x \leq b$ satisfying boundary conditions at the two endpoints a and b ,
 - (i) $k_1 y(a) + k_2 y'(a) = 0$
 - (ii) $l_1 y(b) + l_2 y'(b) = 0$where k_1, k_2, l_1 , and l_2 are constants. We call a solution $y(x)$ an eigenfunction and a number λ for which an eigenfunction exists, an eigenvalue of the problem. Prove the orthogonality of eigenfunctions, y_1, y_2, \dots .

$$\int p(x)y_m(x)y_n(x)dx = 0 \quad \text{for } m \neq n \quad (20 \text{ points})$$

4. What is the Laplace transform? Why is it useful? (10 points)

5. Evaluate $\iint_S (7x \mathbf{i} - z \mathbf{k}) \cdot \mathbf{n} \, dA$ over the sphere $S: x^2 + y^2 + z^2 = 4$ using Two different methods; \mathbf{n} represents the outwardly directed normal unit vector, and \mathbf{i} and \mathbf{k} are the unit vectors in the x and z directions, respectively. (15 points)

6. Find the general solutions that satisfy the following partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right),$$

where c is a constant, and u is a function of time (t) and position (r, θ).

(20 points)

7. Evaluate the real integral (15 points)

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 16}.$$