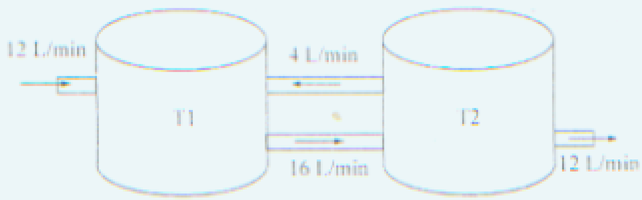
1. Tank T_1 and T_2 initially contain 100 Liter of pure water, respectively. Beginning at time t = 0, the inflow into Tank T_1 from the outside is 12 Liter/min containing 10 g of salt and the outflow to the outside from Tank T_2 is 12 Liter/min, while salt/water solutions are interchanged between the tanks at the flow rates shown in the following figure. The mixtures in tanks are kept uniform by stirring. Determine the amount of salts in each tank at any time t > 0. **(20 points)**



2. (i) Find power series solutions of the following differential equation $(1-x^2)y''-xy'+n^2y=0$, where n is a real number. **(15 points)**

(ii) The above differential equation has a polynomial solution $y = T_n(x)$ for n = 0,1,2,.... Specify the weight function $w(x)$ and the interval over which the set of the polynomials $\{T_n(x)\}$ is orthogonal. Give an orthogonality relation. *(hint: the form of a Sturm-Liouville equation is* $[r(x)y']'+[q(x)+\lambda\, p(x)]y=0$ *)* **(15 points)**

3. A nonlinear ODE $dy/dx = y^3 - 2y + \sin(x)$ with an initial condition $y(0) = 1$ is difficult to find its analytical solution, so a simple numerical method called the Euler method is used to solve it numerically. First, the derivative dy/dx at $x = x_i$ is approximated by a finite-difference formula $(y_{i+1} - y_i)/\Delta x$, where Δx is a small step size, $x_i = (i-1)\,\Delta x$, and $y_i = y(x_i)$. **(25 points)**

   (a) Substituting the approximation formula into the ODE, show how to obtain a series of data (x_i, y_i), $i = 1, \cdots, n$.

   (b) Use the least-squares method to fit the obtained data (x_i, y_i) by the function $\mathbf{y = ae^{bx}}$. What are the optimal parameters **a** and **b**?

4. Consider the following PDE **(25 points)**

$$u_t = a^2 u_{xx} - b\,u \qquad 0 < x < 1 \qquad 0 < t < \infty$$

$$u(0,t) = 0$$

$$u(1,t) = 0$$

$$u(x,0) = \sin(x)$$

   (a) The above PDE can be transformed into a simpler PDE by

   $\mathbf{u(x,t) = e^{-bt}\, w(x,t)}$

   Derive the simper PDE in term of $w(x,t)$ as well as the transformed boundary and initial conditions.

   (b) Solve $w(x,t)$ and then obtain $u(x,t)$.