國立中正大學八十七學年度碩士班考試試題

所　別：化學工程學系　　　　　科　目：工程數學

1. An iterative equation is described by the equation $\mathbf{x}_{n+1} = \mathbf{A}\mathbf{x}_n$, where $\mathbf{A} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$. If $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, compute $\mathbf{x}_{11}$. **(10 points)**

2. (a) Let f(t) have Fourier transform $\phi(\omega)$. Show that
$$\int_{-\infty}^{\infty} |\phi(\omega)|^2 \, d\omega = 2\pi \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$ **(6 points)**

   (b) Use Laplace transform to solve $\frac{d^2x}{dt^2} + \frac{dx}{dt} + 2x = e^{5t}$ with initial conditions $x = 1$ and $\frac{dx}{dt} = 1$ for $t = 0$. **(4 points)**

3. (a) Briefly state the difference between the Laplace and the Fourier Transforms. **(5 points)**

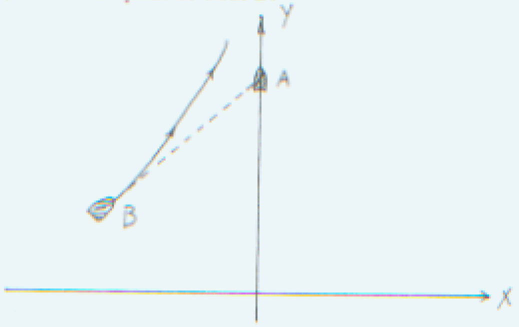
   (b) Briefly state one solution method for an non-homogeneous set of ordinary differential equations $\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{x} \in R^n$. **(5 points)**

   (c) For 4-dimensional vectors **a**, **b**, **c**, and **d**, how to determine whether they can form a basis in the 4-dimensional space. **(5 points)**

4. A boat *A* moves along the y axis with constant speed a. A second boat *B* which moves in the left-hand half of the x-y plane with constant speed b and always points directly at boat *A*.

   (a) Show that the path of boat *B* is described by the differential equation
$$x\frac{d^2y}{dx^2} = -\frac{a}{b}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$ **(9 points)**

   (b) Solve the path of boat *B*. **(6 points)**



5. The time evolution of velocity profile for the Couette flow is described by the partial differential equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \qquad (\nu \text{ is a constant})$$

subject to the following initial and boundary conditions:

$$u(y,0) = 0, \qquad 0 < y < 1,$$
$$u(0,t) = U, \; u(1,t) = 0, \; t > 0.$$

(i) Solve the steady-state velocity profile $u(y)$ (5 points).
(ii) Solve the transient velocity profile $u(t,y)$ (15 points). (Hint: You may use the result in (i) and a change of variable to make the boundary conditions homogeneous)

6. Here we are interested in finding approximate solutions for a first-order, nonlinear, ordinary differential equation,

$$x\frac{dx}{dt} - (1+t)x = \cos(2\pi t), \qquad x(0) = 0, \; x(1) = 1.$$

(i) First show that (5 points)

$$\left.\frac{dx}{dt}\right|_t = \frac{x(t+\Delta t) - x(t-\Delta t)}{2\Delta t} + O\left[(\Delta t)^2\right].$$

(ii) Use the above relation to find an approximate solution for $x(0.5)$ (10 points).

7. Verify the Gauss-Ostrogradskii divergence theorem,

$$\int_V \nabla \cdot \boldsymbol{u}\, dV = \int_S \boldsymbol{n} \cdot \boldsymbol{u}\, dS, \qquad (\boldsymbol{n} \text{ is the outwardly directed normal vector of } dS)$$

for $\boldsymbol{u} = x\vec{i} + y\vec{j} + z\vec{k}$ in the region bounded by $x = \pm 1$, $y = \pm 1$, $z = \pm 1$ (15 points).