1. For a system described by the following equations

$$\frac{dx}{dt}+x+y=e^{-3t} \quad x(0)=1$$

$$\frac{dy}{dt}+4x+y=0 \quad y(0)=0$$

(a) Find the Laplace transform $X(s)$ of $x(t)$. Be sure to eliminate $Y(s)$ from this expression.

(b) Determine the final value of $x(t)$ as $t\to\infty$.

(c) Find the solution $y(t)$. (10分)

2. Find the solution $y(x)$ of the two-point boundary value problem

$$y''-25y=20xe^{-5x},\ y(0)=0,\ y'(1)=1 \qquad (10分)$$

3. The matrix $\mathbf{A}=\begin{bmatrix}-3 & 2\\ 1 & -4\end{bmatrix}$. Evaluate $\exp(\mathbf{A})$ and $\mathbf{A}^{10}$, respectively. (10分)

4. (a) Show that the set $\{1,\ \cos(n\pi x),_{n=1,2,\ldots}\}$ is orthogonal on the interval $[0,\ 1]$.

(b) Determine constants a_0, b_0, b_1, c_0, c_1, and c_2 so that the functions, $f(x)=a_0$ and $g(x)=b_0+b_1x$, and $h(x)=c_0+c_1x+c_2x^2$ form an orthonormal set on the interval $[-1,\ 1]$. (10分)

5. A fluid of constant density ρ is pumped into a cone-shaped tank of total volume $\pi R^2H/3$. The flow out of the bottom of the tank is proportional to the square root of the height $h(t)$ of liquid in the tank. The process is sketched in Figure 1.

(a) Derive the dynamic equations describing the system.

(b) Find the steady state equation for relating the height, h_s, of liquid in the tank and the inlet flow, F_{0s}, at steady state.

(c) Linearize the dynamic equation at the steady state conditions (h_s, F_{0s}). The perturbation variables are defined as $\Delta h(t)=h(t)-h_s$, and $\Delta F_0(t)=F_0(t)-F_{0s}$.

(d) Find the transfer function, $\Delta h(s)/\Delta F_0(s)$, of the linearized system by using Laplace transform.

(e) Find the perturbation solution $\Delta h(t)$ while the inlet flow rate $F_0(t)$ at $t=0$ is suddenly changed from the value F_{0s} at steady state to $2F_{0s}$. (10分)

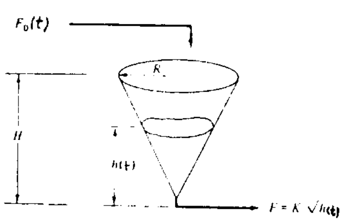

Figure 1

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6. (a) What is a partial differential equation (PDE) ? (3分)
   (b) What is the importance of PDE in the study of Chemical Engineering? (5分)
   (c) Write down a PDE which describes a chemical process/system. Explain the PDE you wrote down. (6分)

7. (a) Show that

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = \frac{\pi}{\sqrt{2}} \qquad (6分)$$

   (b) 2-dimensional Laplacian in rectangular coordinate (x,y) is

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

   Show that

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{in cylindrical coordinate } (r,\theta). \qquad (6分)$$

   (Note: $x = r\cos\theta$ and $y = r\sin\theta$)

8. Consider the following nonlinear differential equation

$$\frac{dy}{dt} + y^2 = y\sin t$$

   with the initial condition $y(0) = 0.1$, which is difficult, or even impossible, to be solved analytically. Numerical method such as finite-difference method can be used to solve the above equation.

   (a) Show that

$$\left.\frac{dy}{dt}\right|_{t=t_i} \text{ can be approximated by } \frac{y(t_i+\Delta t) - y(t_i)}{\Delta t}, \text{ where } \Delta t \text{ is the time step.} \qquad (4分)$$

   (b) If $\Delta t = 0.1$, compute values of $y(0.1)$ and $y(0.2)$. (8分)

   (Note: $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.1987$)

9. A moving boundary problem is described by the following differential equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 \le x \le 1, \quad 0 < t < \infty \qquad (9a)$$

   with nonhomogeneous boundary conditions

$$u(0,t) = g_1(t), \quad 0 < t < \infty \qquad (9b)$$

$$u(1,t) = g_2(t), \quad 0 < t < \infty \qquad (9c)$$

   and an initial condition

$$u(x,0) = f(x), \quad 0 \le x \le 1 \qquad (9d)$$

   Equations (9a-d) can be transformed into the following differential equation

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - \frac{\partial s}{\partial t} \qquad (9e)$$

   with **homogeneous** boundary conditions

$$v(0,t) = 0 \qquad (9f)$$

$$v(1,t) = 0 \qquad (9g)$$

   and an initial condition

$$v(x,0) = f(x) - s(x,0) \qquad (9h)$$

   by using the transformation

$$u(x,t) = s(x,t) + v(x,t) = a(t)(1-x) + b(t)x + v(x,t)$$

   (a) determine $a(t)$ and $b(t)$. (4分)

   (b) Equations (9e-h) can then be solved by the method of eigenfunction expansions.

   Let $v(x,t) = \sum_{n=1}^{\infty} c_n(t)\sin(n\pi x)$ and solve equations (9e-h). (8分)

   (Note: $\sin(n\pi x)$, $n = 1, \cdots, \infty$, are eigenfunctions)