1. Reversible reactions, $A + A \longleftrightarrow B$, taking place in a CSTR can be described by the following differential equations:

$$V\frac{da}{dt} = Q(a_0 - a) - Vk_1a^2 + Vk_2b \tag{1a}$$
$$V\frac{db}{dt} = -Qb + Vk_1a^2 - Vk_2b \tag{1b}$$

where V is the reactor volume, Q is the flow rate, k_1 and k_2 are rate constants, a_0 is the input concentration of A, and a and b are concentrations of A and B, respectively.

(a) Show that Equations (1a) and (1b) can be transformed into the following dimensionless equations: (3分)

$$\frac{dx_1}{d\tau} = 1 - x_1 - \alpha_1x_1^2 + \alpha_2x_2 \tag{1c}$$
$$\frac{dx_2}{d\tau} = -x_2 + \alpha_1x_1^2 - \alpha_2x_2 \tag{1d}$$

by introducing the dimensionless variables $x_1 = a/a_0, x_2 = b/a_0, \tau = tQ/V, \alpha_1 = Vk_1a_0/Q$, and $\alpha_2 = Vk_2/Q$.

(b) Let the deviation equation of Eqs. (1c) and (1d) be defined by

$$\frac{d\underline{x}}{d\tau} = \underline{J}\,\underline{x}, \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \underline{J} = \begin{bmatrix} -1-2\alpha_1 & \alpha_2 \\ 2\alpha_1 & -1-\alpha_2 \end{bmatrix} \tag{1e}$$

Prove that the system described by Eq. (1e) is stable. (3分)
(Note: A system is said to be stable if all eigenvalues of the system matrix have negative real parts)

(c) For $\alpha_1 = 1$ and $\alpha_2 = 2$, the matrix $\underline{J}$ in Eq. (1e) is symmetric and its two eigenvectors are orthogonal to each other. Show that if λ_i and λ_j are two different eigenvalues of a real symmetric matrix $\underline{A}$, then the associated eigenvectors $\underline{u}_i$ and $\underline{u}_j$ are othogonal to each other, i.e. $\underline{u}_i \cdot \underline{u}_j = 0$. (6分)

2. Let $(t_i, x_i), i = 1, 2, \cdots, N$, be a sequence of data recorded from an experiment. Find the optimal parameters α and β in the linear model $x(t) = \alpha t + \beta$ such that the sum of squared errors, $J = \sum_{i=1}^{N}(x_i - x(t_i))^2$ is minimized.
(Hint: the optimal α and β can be found by letting $\partial J/\partial\alpha = \partial J/\partial\beta = 0$) (5分)

3. (a) Find the solution $y(x)$ of the initial-value problem (7分)

$$(x+a)^2\frac{d^2y}{dx^2} - 4(x+a)\frac{dy}{dx} + 6y = 0, \qquad y(0) = 0, \quad \frac{dy(0)}{dx} = 1 \tag{3}$$

(b) Find the maximum of $y(x)$. (3分)

4. Consider the ordinary differential equation

$$\frac{d\underline{x}}{dt} = \underline{J}\,\underline{x}, \qquad \underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \underline{J} = \begin{bmatrix} 2 & 1 & 2 \\ -1 & 0 & -2 \\ k & 0 & 1 \end{bmatrix} \tag{4}$$

(a) Find the eigenvalues of $\underline{J}$. (2分)

(b) If $k = 0$, find the eigenvalues and the associated eigenvectors of $\underline{J}$, and then find the general solution of the differential equation in Eq. (4). (6分)

5. (a) The continuity equation for fluid flow is given by

$$\frac{\partial\rho}{\partial t} = -(\nabla\cdot\rho\underline{v}), \qquad \underline{v} = \begin{pmatrix} u \\ v \end{pmatrix} \tag{5}$$

(續下頁)

國立中正大學八十三學年度碩士班考試試題

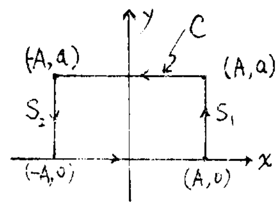
所　別：化學工程研究所　　　　科　目：工 程 數 學　　　　

Show that a stream function $\Psi(x, y)$ can be defined such that $\partial u/\partial x = -\partial\Psi/\partial y$ and $\partial v/\partial y = \partial\Psi/\partial x$ for two-dimensional incompressible flow (ρ is constant). (5分)

(b) What are Stokes' and Divergence theorems ? What are their importance in the study of chemical engineering ? (10分)

6. Consider the integral of $f(z) = e^{-z^2}$ over a closed rectangular path C shown in Fig. 1.

Fig. 1.



(a) Show that (3分)

$$\int_{-A}^{+A} e^{-x^2}\,dx - \int_{-A}^{+A} e^{-(x+ia)^2}\,dx + \int_{S_1} e^{-z^2}\,dz + \int_{S_2} e^{-z^2}\,dz = 0, \qquad i = \sqrt{-1} \tag{6a}$$

where S_1 is the line segment from $(A, 0)$ to (A, a) and S_2 is the line segment from $(-A, a)$ to $(-A, 0)$.

(b) Show that, on both S_1 and S_2, (3分)

$$|e^{-z^2}| = e^{-(A^2-y^2)} \le e^{-(A^2-a^2)} \tag{6b}$$

Note that the lengths of S_1 and S_2 are equal to a.

(c) Show that the integrals $\int_{S_1} e^{-z^2}dz$ and $\int_{S_2} e^{-z^2}dz$ tend to zero as $A \to \infty$ for any fixed value of a. (3分)

(d) Show that (3分)

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-(x+ia)^2}\,dx \tag{6c}$$

7. (a) Show that (4分)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \pi \tag{7a}$$

(b) Show that (3分)

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} \tag{7b}$$

(c) The Fourier transform of a function $f(t)$ is defined by $\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$, $i = \sqrt{-1}$. Find the Fourier transform of $e^{-t^2/2}$ (3分)

8. The gamma function $\Gamma(\alpha)$, $\alpha > 0$, is defined by the integral

$$\Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha-1}e^{-t}\,dt \tag{8}$$

(a) Show that $\Gamma(k+1) = k!$, $k = 0, 1, 2, \cdots$. (4分)

(b) Show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. (4分)

(Note: You can use the results of previous problems)

9. Use the Laplace transform to find the solution $z(x, t)$ of the problem (10分)

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = z, \qquad z(0, t) = 1, \quad z(x, 0) = 1 \tag{9}$$

where $0 \le x < \infty$ and $0 \le t < \infty$.